Which Disagreements Last the Longest?

Due: Wednesday, September 23 at 11:59pm

Instructions: Please submit your *handwritten* solutions as a single PDF on Gradescope.

People adjust their opinions toward those of their friends. Even as everyone moves toward agreement, two groups can remain on opposite sides of the average.

Consider n people in a connected friendship graph, each with d friends. Friendships are mutual. At each round, everyone simultaneously replaces their opinion with the average of their d friends' opinions from the previous round: add these d opinions and divide by d. Let $\mathbf{x}^{(k)} \in \mathbb{R}^n$ contain the opinions after k rounds.

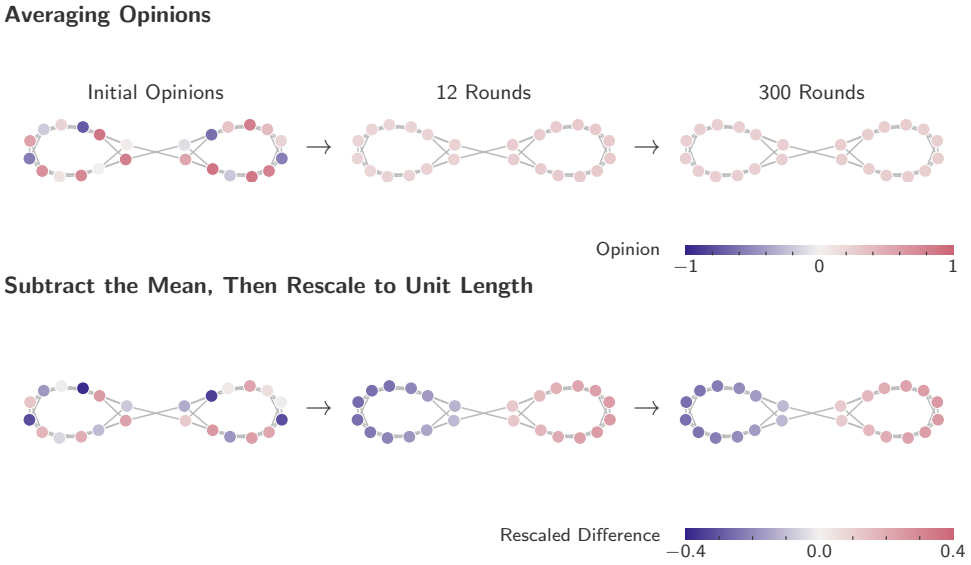
1. Each round averages opinions across the friendship graph. What matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ makes this update $\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)}$?

2. The initial mean is $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i^{(0)}$. Prove that the mean opinion remains μ after every round.

3. To focus on relative opinions, subtract the initial mean and rescale:

$$\mathbf{y}^{(k)} = \frac{\mathbf{x}^{(k)} - \mu\mathbf{1}}{\|\mathbf{x}^{(k)} - \mu\mathbf{1}\|},$$

where $\mathbf{1}$ is the all-ones vector. Using the power method and the hint below, find the limit of $\mathbf{y}^{(k)}$ in terms of the starting opinions, assuming $\mathbf{v}_2^\top \mathbf{x}^{(0)} \neq 0$.



The same people stay in the same positions. Rescaling reveals the pattern of remaining disagreement, even while its size shrinks.

Hint: Assume $\mathbf{A}$ has an orthonormal eigenbasis with

$$\mathbf{v}_1 = \frac{\mathbf{1}}{\sqrt{n}}, \qquad 1 = \lambda_1 > \lambda_2 > |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Takeaway: *Everyone can move toward agreement while the same groups stay above and below the average. The graph's second eigenvector describes the disagreement that lasts longest.*