# How Should We Compute a Matrix Power Times a Vector?

**Due:** Wednesday, September 16 at 11:59pm

**Instructions:** Please submit your *handwritten* solutions as a single PDF on Gradescope.

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$, let $\mathbf{x} \in \mathbb{R}^d$, and let $q \geq 2$ be an integer. We want to compute $\mathbf{A}^q\mathbf{x}$ using standard dense matrix arithmetic. Forming $\mathbf{A}^q$ through $q-1$ dense matrix products and then applying it costs $O(qd^3)$ time.

1. Suppose we compute

$$\mathbf{A}^q\mathbf{x} = \mathbf{A}\big(\mathbf{A}(\cdots(\mathbf{A}\mathbf{x}))\big)$$

   by applying $\mathbf{A}$ to a vector $q$ times. What is the time complexity?

2. We can instead compute $\mathbf{A}^q\mathbf{x}$ in $O(d^3 \log_2 q)$ time. Explain how to achieve this bound using repeated squaring and the binary expansion of $q$.

3. Suppose the rank-one factors are given, so $\mathbf{A} = \mathbf{u}\mathbf{v}^\top$ for nonzero $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. You may use

$$\mathbf{A}^q\mathbf{x} = (\mathbf{v}^\top\mathbf{u})^{q-1}\mathbf{u}(\mathbf{v}^\top\mathbf{x}).$$

   What time complexity can you achieve for computing $\mathbf{A}^q\mathbf{x}$?

**Hint:** A length-$d$ inner product takes $O(d)$ time. You may write $q$ as a sum of distinct powers of two.

***Takeaway:*** *The same matrix expression can take very different amounts of work to compute. Changing the order of multiplication, reusing powers, or using a rank-one factorization can avoid many unnecessary calculations.*