When Do a Few Rows and Columns Determine a Matrix?

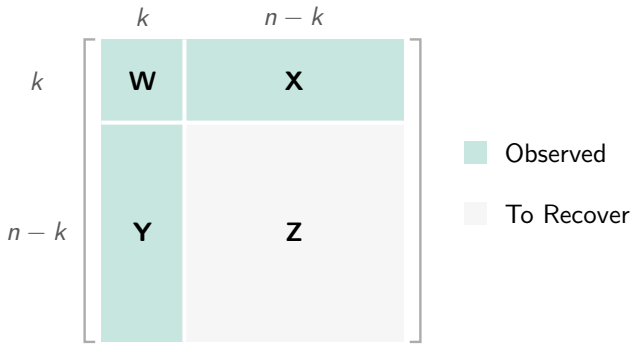
Due: Wednesday, November 4 at 11:59pm

Instructions: Please submit your *handwritten* solutions as a single PDF on Gradescope.

A matrix with millions of entries can sometimes be recovered from just a few complete rows and columns. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ have rank k, where $1 \leq k < n$, and partition it as

$$\mathbf{A} = \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix}.$$

We observe $\mathbf{W} \in \mathbb{R}^{k \times k}$, $\mathbf{X} \in \mathbb{R}^{k \times (n-k)}$, and $\mathbf{Y} \in \mathbb{R}^{(n-k) \times k}$. Assume $\text{rank}(\mathbf{W}) = k$. The remaining block $\mathbf{Z} \in \mathbb{R}^{(n-k) \times (n-k)}$ is unknown.



1. The observed block $\mathbf{W}$ has k independent columns. Why do the first k columns of $\mathbf{A}$ span its entire column space?
2. Each remaining column is a linear combination of those first k columns. How can we recover the missing block $\mathbf{Z}$ from $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$?
3. We store only the observed rows and columns. What fraction of the n^2 entries do we need to store?

Takeaway: *Low rank links the missing entries to the observed ones. When the observed intersection has full rank, a few complete rows and columns determine the entire matrix, even as its dimensions grow.*