CSCI 145 Problem 1 Solution: The Correlation Floor

Solution

Approach

Every distinct pair contributes the same covariance. Counting those contributions gives the exact variance and its limit.

1. Variance

Because every response has variance σ^2 and every distinct pair has covariance $\rho\sigma^2$,

$$\begin{aligned}\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \operatorname{Cov}(X_i, X_j) \\ &= n\sigma^2 + 2\binom{n}{2}\rho\sigma^2 \\ &= n\sigma^2 + n(n-1)\rho\sigma^2.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Var}(\bar{X}) &= \frac{\sigma^2}{n^2}\left[n + n(n-1)\rho\right] \\ &= \sigma^2\left(\rho + \frac{1-\rho}{n}\right),\end{aligned}$$

so for fixed ρ,

$$\operatorname{Var}(\bar{X}) \longrightarrow \rho\sigma^2.$$

2. The Analyst's Claim

For a fixed positive ρ, increasing n removes the term $(1-\rho)\sigma^2/n$, but the floor $\rho\sigma^2$ remains. The analyst's claim is therefore false, and reducing ρ lowers the floor.

At $\rho = 0$, the familiar independent result is recovered:

$$\operatorname{Var}(\bar{X}) = \frac{\sigma^2}{n} \longrightarrow 0.$$

At $\rho = 1$,

$$\operatorname{Var}(\bar{X}) = \sigma^2,$$

so perfectly correlated responses contain no more averaging information than one response.