# CSCI 145 Problem 4 Solution: Opinions and Polarization

## 1. The Averaging Matrix

Person $i$ averages the opinions of their $d$ friends. Each friend's opinion receives weight $1/d$, so

$$A_{i,j} = \begin{cases} \frac{1}{d} & i \text{ and } j \text{ are friends,} \\ 0 & \text{otherwise,} \end{cases}$$

The resulting update is

$$x_i^{(k+1)} = \frac{1}{d} \sum_{j:\, j \text{ is a friend of } i} x_j^{(k)}.$$

The coordinate update is exactly $\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)}$. Each row sums to 1, so $\mathbf{A}\mathbf{1} = \mathbf{1}$: equal opinions stay equal. Mutual friendships make $\mathbf{A}$ symmetric, so its columns also sum to 1 and the mean is preserved.

## 2. The Limiting Direction of Disagreement

Expand the starting opinions in the given eigenbasis:

$$\mathbf{x}^{(0)} = \sum_{i=1}^{n} c_i \mathbf{v}_i, \qquad c_i = \mathbf{v}_i^\top \mathbf{x}^{(0)}.$$

Since $c_1 = \sqrt{n}\,\mu$ and $\lambda_1 = 1$, the reading's formula for $\mathbf{A}^k$ gives

$$\mathbf{x}^{(k)} = \mu\mathbf{1} + \sum_{i=2}^{n} c_i \lambda_i^k \mathbf{v}_i.$$

All terms in the sum decay, so the raw opinions converge to consensus $\mu\mathbf{1}$. Subtracting the mean removes exactly the top eigenvector's contribution. Because $c_2 \neq 0$ and $\lambda_2 > 0$, we can write

$$\mathbf{x}^{(k)} - \mu\mathbf{1} = c_2 \lambda_2^k (\mathbf{v}_2 + \mathbf{r}^{(k)}), \qquad \mathbf{r}^{(k)} = \sum_{i=3}^{n} \frac{c_i}{c_2} \left(\frac{\lambda_i}{\lambda_2}\right)^k \mathbf{v}_i.$$

Every ratio has magnitude less than 1, so $\mathbf{r}^{(k)} \to \mathbf{0}$. This is the power method with the constant direction removed. Normalizing gives

$$\mathbf{y}^{(k)} = \frac{c_2}{|c_2|} \frac{\mathbf{v}_2 + \mathbf{r}^{(k)}}{\|\mathbf{v}_2 + \mathbf{r}^{(k)}\|} \longrightarrow \operatorname{sign}(c_2)\mathbf{v}_2.$$

The network determines the limiting shape of disagreement. Starting opinions can reverse its sign, exchanging which group is above the mean, but otherwise do not change that shape when $c_2 \neq 0$. They also affect the consensus value and how quickly the limiting shape becomes visible.

In a network with two closely connected groups and few friendships between them, this direction can separate the groups, as the figure shows. The model illustrates how social connections can organize relative opinions into opposing camps, a possible ingredient in perceived political polarization. The absolute differences still shrink. Rescaling makes their pattern visible.