CSCI 145 Problem 5 Solution: Can We Run a Fitted Line Backward?

Solution

Geometric argument suggested by Suchen.

1. The Two Fitted Rules

Write $\bar{x} = \frac{1}{n}\sum_i x_i$ and $\bar{y} = \frac{1}{n}\sum_i y_i$. The reading's formulas give Engineer A's optimal parameters:

$$w = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}, \qquad b = \bar{y} - w\bar{x}.$$

Exchanging the feature and label gives Engineer B's optimal parameters:

$$a = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (y_i - \bar{y})^2}, \qquad c = \bar{x} - a\bar{y}.$$

Both denominators are positive because both coordinates vary. The resulting prediction rules are

$$\widehat{x}_A(y) = \frac{y - b}{w} = \bar{x} + \frac{y - \bar{y}}{w}, \qquad \widehat{x}_B(y) = ay + c = \bar{x} + a(y - \bar{y}).$$

2. When the Rules Agree

They pass through the same sample-average point, so they agree for every y exactly when $a = 1/w$.

Use the centered vectors from the hint, writing $\mathbf{x}_c = \mathbf{x}_{\text{centered}}$ and $\mathbf{y}_c = \mathbf{y}_{\text{centered}}$ for short:

$$(\mathbf{x}_c)_i = x_i - \bar{x}, \qquad (\mathbf{y}_c)_i = y_i - \bar{y}.$$

Both are nonzero because both coordinates vary. The slope formulas become

$$w = \frac{\langle \mathbf{x}_c, \mathbf{y}_c \rangle}{\|\mathbf{x}_c\|_2^2}, \qquad a = \frac{\langle \mathbf{x}_c, \mathbf{y}_c \rangle}{\|\mathbf{y}_c\|_2^2}.$$

Let θ be the angle between these vectors. Multiplying the slopes and applying the inner-product identity gives

$$aw = \frac{\langle \mathbf{x}_c, \mathbf{y}_c \rangle^2}{\|\mathbf{x}_c\|_2^2 \|\mathbf{y}_c\|_2^2} = \cos^2\theta.$$

Thus $a = 1/w$ exactly when $aw = 1$, or $\cos\theta = 1$ or -1. The centered vectors must point in the same or opposite directions: $\mathbf{y}_c = t\mathbf{x}_c$ for some nonzero scalar t. In coordinates, this means

$$y_i - \bar{y} = t(x_i - \bar{x}) \quad \Longleftrightarrow \quad y_i = tx_i + (\bar{y} - t\bar{x}) \quad \text{for every } i.$$

All measurements therefore lie on one nonhorizontal straight line. Conversely, measurements on such a line have proportional centered vectors, so $\cos^2\theta = 1$, $aw = 1$, and the two prediction rules agree.