CSCI 145 Problem 15 Solution: When Do a Few Rows and Columns Determine a Matrix?

Solution

1.

The k columns of $\mathbf{W}$ have rank k, so they form a basis of $\mathbb{R}^k$ and $\mathbf{W}$ is invertible. Let $\mathbf{C} \in \mathbb{R}^{n \times k}$ contain the first k columns of $\mathbf{A}$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{W} \\ \mathbf{Y} \end{bmatrix}.$$

If $\mathbf{Ca} = \mathbf{0}$ for $\mathbf{a} \in \mathbb{R}^k$, its first k coordinates give $\mathbf{Wa} = \mathbf{0}$, so $\mathbf{a} = \mathbf{0}$. Thus these k columns are linearly independent. Since $\operatorname{rank}(\mathbf{A}) = k$, they form a basis of the entire column space of $\mathbf{A}$.

2.

Every remaining column is a linear combination of the columns of $\mathbf{C}$. Collecting their coefficients in $\mathbf{H} \in \mathbb{R}^{k \times (n-k)}$ gives

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{W} \\ \mathbf{Y} \end{bmatrix} \mathbf{H}.$$

The top block determines the coefficients: $\mathbf{X} = \mathbf{WH}$, hence $\mathbf{H} = \mathbf{W}^{-1}\mathbf{X}$. Substituting into the bottom block gives

$$\mathbf{Z} = \mathbf{YW}^{-1}\mathbf{X}.$$

The observed top block uniquely determines the coefficients, which then determine every entry of $\mathbf{Z}$.

3.

The observed blocks contain

$$k^2 + k(n-k) + (n-k)k = 2nk - k^2$$

entries. The overlap $\mathbf{W}$ is counted once. As a fraction of all n^2 entries, this is

$$\frac{2nk - k^2}{n^2} = \frac{2k}{n} - \left(\frac{k}{n}\right)^2.$$

For fixed k, the observed count grows linearly in n, while the full matrix grows quadratically. The observed fraction tends to zero, even though it determines all $(n-k)^2$ unobserved entries exactly.