CSCI 145 Problem 2: The Optimal Control Variate

August 27, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on DATE**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- Problems are grouped into three-week **cycles**. Once per cycle, you will meet with the grader and present the solution to *one* problem from that cycle, selected at random when you arrive — so bring your work and be ready to explain any of them.

Problem 2: The Optimal Control Variate 2 points

We want $\mu = \mathbb{E}[f(X)]$ and know the exact mean of a surrogate $g(X)$, where $\mathrm{Var}(f(X)) > 0$ and $\mathrm{Var}(g(X)) > 0$. From independent samples $X^{(1)}, \dots, X^{(n)}$, lecture constructed the unbiased family

$$\hat{\mu}_c = \frac{1}{n}\sum_{i=1}^{n}\Big[f(X^{(i)}) - c\big(g(X^{(i)}) - \mathbb{E}[g(X)]\big)\Big].$$

Determine the best member of this family. A complete analysis should find the coefficient $c^\star$ that minimizes its variance, express the minimum variance as a multiple of $\mathrm{Var}(f(X))/n$ using the correlation ρ between $f(X)$ and $g(X)$, and explain what your formulas predict when the surrogate is uncorrelated with the target.

Test the result on the lecture's dice example, where f is the larger of two dice and g is their sum. Enumeration of the 36 outcomes gives

$$\mathrm{Var}(f(X)) = \frac{2555}{1296}, \qquad \mathrm{Cov}(f(X), g(X)) = \frac{35}{12}, \qquad \mathrm{Var}(g(X)) = \frac{35}{6}.$$

Compute $c^\star$, compare it with lecture's choice $c = \frac{1}{2}$, and determine how many independent games the unadjusted estimator would need to match the variance of the optimally adjusted estimator using n games.

Results you may use. The estimator above is unbiased for every constant c; the variance of an average of n independent, identically distributed random variables is the variance of one divided by n; and $\mathrm{Var}(U + aV) = \mathrm{Var}(U) + 2a\,\mathrm{Cov}(U,V) + a^2\,\mathrm{Var}(V)$. A quadratic $ac^2 + bc + d$ with $a > 0$ is minimized at $c = -b/(2a)$.