CSCI 145 Problem 3: The Cost of a Matrix Power

October 27, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on Wednesday, September 16**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- A complete, good-faith attempt earns submission credit. The grader may select any recent assigned problem for a brief conversation, so keep your work and be prepared to explain each solution and answer a related question. Scheduling details will be announced in class.

Problem 3: The Cost of a Matrix Power

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$, let $\mathbf{x} \in \mathbb{R}^d$, and let $q \geq 2$ be an integer. We want to compute $\mathbf{A}^q\mathbf{x}$ using standard dense matrix arithmetic.

1. Suppose you first form $\mathbf{A}^q$ through the sequence $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A}$, and so on, and then multiply the result by $\mathbf{x}$. What is the time complexity?

2. Instead, compute

$$\mathbf{A}^q\mathbf{x} = \mathbf{A}\big(\mathbf{A}(\cdots(\mathbf{A}\mathbf{x}))\big)$$

   by applying $\mathbf{A}$ to a vector q times. What is the time complexity?

3. We can instead compute $\mathbf{A}^q\mathbf{x}$ in $O(d^3 \log_2 q)$ time. Explain how to achieve this bound using repeated squaring and the binary expansion of q.

4. Suppose the rank-one factors are given, so $\mathbf{A} = \mathbf{u}\mathbf{v}^\top$ for nonzero $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. What is the best time complexity you can achieve for computing $\mathbf{A}^q\mathbf{x}$? Give an expression for $\mathbf{A}^q\mathbf{x}$ that leads to this bound.

Hint: A length-d inner product takes $O(d)$ time. You may write q as a sum of distinct powers of two.

Takeaway: *The same matrix expression can take very different amounts of work to compute. Changing the order of multiplication, reusing powers, or using a rank-one factorization can avoid many unnecessary calculations.*