CSCI 145 Problem 14: Does Gradient Descent Implicitly Regularize?

October 27, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on Wednesday, October 28**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- A complete, good-faith attempt earns submission credit. The grader may select any recent assigned problem for a brief conversation, so keep your work and be prepared to explain each solution and answer a related question. Scheduling details will be announced in class.

Problem 14: Does Gradient Descent Implicitly Regularize?

Many weight vectors can make the same training predictions. For inputs $\mathbf{x}_i \in \mathbb{R}^d$ and labels $y_i \in \mathbb{R}$, train $f_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$ by minimizing

$$\mathcal{L}(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^{n} \big(\mathbf{w}^\top \mathbf{x}_i - y_i\big)^2.$$

Starting at $\mathbf{w}^{(0)} = \mathbf{0}$, apply gradient descent with learning rate $\alpha > 0$:

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \alpha \nabla \mathcal{L}(\mathbf{w}^{(t)}).$$

1. Prove that $\mathbf{w}^{(t)} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for every integer $t \geq 0$.

2. Prove that every iterate has the smallest Euclidean norm among all weights with the same training predictions. Specifically, for every t and every $\mathbf{v} \in \mathbb{R}^d$, prove

$$\mathbf{v}^\top \mathbf{x}_i = (\mathbf{w}^{(t)})^\top \mathbf{x}_i \text{ for all } i \quad \Longrightarrow \quad \|\mathbf{w}^{(t)}\|_2 \leq \|\mathbf{v}\|_2.$$

   Hint: Write $\mathbf{v} = \mathbf{w}^{(t)} + \mathbf{z}$ and use the matching predictions to show that $\mathbf{z}$ is orthogonal to the input span.

Takeaway: *For this linear model, gradient descent started at zero always finds weights with the smallest norm among those giving the same training predictions. If it reaches a perfect fit, that fit has the smallest weight norm. Gradient descent can favor small weights without an explicit penalty.*