CSCI 145 Problem 15: Compress for the Deployment

August 27, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on DATE**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- Problems are grouped into three-week **cycles**. Once per cycle, you will meet with the grader and present the solution to *one* problem from that cycle, selected at random when you arrive; bring your work and be ready to explain any of them.

Problem 15: Compress for the Deployment 2 points

A linear layer $\mathbf{A} \in \mathbb{R}^{m\times n}$ must be replaced by a matrix $\mathbf{B}$ of rank at most k. Two deployment teams disagree about the objective. The safety team minimizes the worst-case error

$$\sup_{\|\mathbf{x}\|_2=1} \|(\mathbf{A}-\mathbf{B})\mathbf{x}\|_2,$$

so it uses the truncated SVD $\mathbf{A}_k$. The traffic team instead wants to minimize the average error $\mathbb{E}\|(\mathbf{A}-\mathbf{B})\mathbf{x}\|_2^2$ for inputs with mean zero and positive-definite covariance $\mathbf{\Sigma}_x$.

Determine and justify an optimal rank-k layer for the traffic team. Your argument should reduce its objective to a low-rank approximation problem to which Eckart–Young–Mirsky applies, and it should make clear why the resulting layer need not equal $\mathbf{A}_k$.

Give an explicit 2×2 matrix $\mathbf{A}$, positive-definite covariance $\mathbf{\Sigma}_x$, and $k=1$ for which the two teams choose different compressed layers. Use the example to explain how deployment frequency can outweigh a direction's worst-case error.

Results you may use. You may quote the worst-case result $\mathbf{A}_k \in \arg\min_{\operatorname{rank}(\mathbf{B})\le k} \|\mathbf{A}-\mathbf{B}\|_2$, Eckart–Young–Mirsky in Frobenius norm, and the identity

$$\mathbb{E}\|(\mathbf{A}-\mathbf{B})\mathbf{x}\|_2^2 = \|(\mathbf{A}-\mathbf{B})\mathbf{\Sigma}_x^{1/2}\|_F^2,$$

which holds when $\mathbb{E}[\mathbf{x}]=\mathbf{0}$ and $\mathbb{E}[\mathbf{x}\mathbf{x}^\top]=\mathbf{\Sigma}_x$. Because $\mathbf{\Sigma}_x$ is positive definite, $\mathbf{\Sigma}_x^{1/2}$ is invertible and $\operatorname{rank}(\mathbf{B}\mathbf{\Sigma}_x^{1/2}) = \operatorname{rank}(\mathbf{B})$.