CSCI 145 Problem 18: Recover the Muon Cubic

August 27, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on DATE**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- Problems are grouped into three-week **cycles**. Once per cycle, you will meet with the grader and present the solution to *one* problem from that cycle, selected at random when you arrive; bring your work and be ready to explain any of them.

Problem 18: Recover the Muon Cubic 2 points

Suppose a matrix iteration acts on each singular value through an odd cubic

$$p(x) = ax + bx^3.$$

The design brief requires that 0 and 1 be fixed points, that p map $[0, 1]$ monotonically into itself, and that every $x \in (0, 1)$ move strictly closer to 1. Among all cubics satisfying the brief, choose the one that makes the fastest possible initial progress on very small singular values, as measured by $p'(0)$.

Determine whether the optimizer is unique and prove that it satisfies every requirement on the full interval, not only near the endpoints. Translate the winning scalar map into an iteration involving a matrix $\mathbf{X}_t$ and its transpose. Finally, assess whether the familiar coefficient $3/2$ in this iteration is an arbitrary convention or is forced by this design brief, and identify which requirements fail if the coefficient is increased to obtain faster small-singular-value growth.

Results you may use. For $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$, an update of the form $a\mathbf{X} + b\mathbf{X}\mathbf{X}^\top\mathbf{X}$ replaces each singular value s by $as + bs^3$. Monotonicity together with the endpoint values is enough to keep a continuous scalar map inside $[0, 1]$.