CSCI 145 Problem 19: Does Equivariance Force Convolution?

November 4, 2026

Submission Instructions

Please upload your work to Gradescope by **11:59pm on Wednesday, November 25**.

- Turn in a single PDF of *handwritten* solutions. Typed (including LaTeX) solutions will not be accepted.
- A complete, good-faith attempt earns submission credit. The grader may select any recent assigned problem for a brief conversation, so keep your work and be prepared to explain each solution and answer a related question. Scheduling details will be announced in class.

Problem 19: Does Equivariance Force Convolution?

A circular array of n environmental sensors records $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{S}$ shift a signal by one sensor, and let the linear map $\mathbf{T} \in \mathbb{R}^{n \times n}$ be a detector whose output must shift with its input:

$$\mathbf{TS} = \mathbf{ST}.$$

Weight sharing makes a convolution equivariant, but the engineering team wants to know whether an entirely different linear construction could satisfy the same requirement.

1. Why are the linear detectors that commute with the circular shift exactly the maps with coefficients $w_0, \ldots, w_{n-1}$ of the form

   $$\mathbf{T} = \sum_{j=0}^{n-1} w_j \mathbf{S}^j$$

   ?

2. If each output may depend only on sensors within circular distance h, where $2h+1 \leq n$, what form does the resulting convolution take, and how many trainable scalars remain?

Hint: For the standard basis indexed modulo n, you may use $\mathbf{e}_j = \mathbf{S}^j \mathbf{e}_0$.

Takeaway: *For a linear detector on a circular array, requiring outputs to shift with inputs forces the detector to be a convolution. Requiring it to use only nearby sensors also limits the number of weights. Weight sharing follows from the behavior we want the detector to have.*